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AN EXISTENCE THEOREM FOR THE DIRICHLET
INITIAL-BOUNDARY VALUE PROBLEM IN
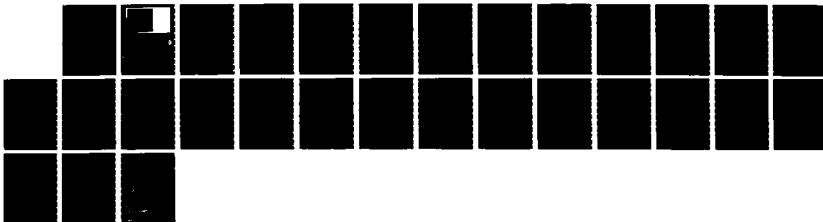INCOMPRESSIBLE NONLINEAR ELASTICITY

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IN INCOMPRESSIBLE NONLINEAR ELASTICITY

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ABSTRACT

We apply energy methods to prove an existence theorem for the Dirichlet initial-boundary value problem in incompressible nonlinear elasticity.

AMS (MOS) Subject Classifications: 35L70, 35M05, 73C50

Key Words: Incompressible elasticity, initial-boundary value problem, energy methods

Work Unit Number 1 (Applied Analysis)

SIGNIFICANCE AND EXPLANATION

Many elastic materials, such as rubber, can be subjected to large volume-preserving deformations, but show little compressibility. Such materials are often modelled as incompressible elastic solids. In this paper, the authors study the initial-boundary value problem for the differential equations describing such materials and prove that it is well-posed.

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AN EXISTENCE THEOREM FOR THE DIRICHLET

INITIAL-BOUNDARY VALUE PROBLEM

IN INCOMPRESSIBLE NONLINEAR ELASTICITY

William J. Hrusa[1] and Michael Renardy[2]

1. Introduction and statement of results

The goal of this paper is to show how energy methods can be applied to establish the local (in time) existence of classical solutions to the Dirichlet initial-boundary value problem in incompressible nonlinear elasticity. For compressible nonlinear elasticity, the initial value problem posed on all of space was solved by Hughes, Kato and Marsden [7] using methods of semigroup theory, and the Dirichlet initial-boundary value problem was solved by Kato [8], Chen and von Wahl [3] and Dafermos and Hrusa [4]. Schochet [14] proves an existence theorem for the initial value problem in the incompressible case by considering incompressible materials as a limit of compressible materials and deriving uniform estimates which allow passing to the limit. His proofs are based on ideas of Klainerman and Majda [9,10], who considered the incompressible limit in gas dynamics. A direct existence proof for incompressible elasticity (without using approximation by compressible materials) is given by Ebin and Saxton [5]. The results of Renardy [13] can also be applied to incompressible elasticity. In this way, an existence theorem for the incompressible initial value problem is obtained, but it is not proved that solutions of the incompressible problem are limits of solutions for the compressible case. To our knowledge, there are no previous results on initial-boundary value problems for incompressible elasticity.

In all the papers quoted above, energy estimates play an essential role. A first order energy estimate can always be obtained by considering the physical energy of the elastic body, which is the integral of an expression involving first derivatives of the unknown deformation function. Bounds on this energy, however, are not strong enough to be useful in an existence proof. The idea is to differentiate the equations of motion and consider the analogues of the energy for higher derivatives of the solution. The a priori bounds thus obtained are then used in a contraction argument establishing the existence of a solution. There is an essential difference here between the initial value problem on all of space and

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the initial-boundary value problem. For the initial-value problem one may differentiate the equations with respect to spatial variables as well as time. For initial-boundary value problems, however, differentiation with respect to spatial variables "destroys" the boundary conditions; in other words, we do not know anything about spatial derivatives of the solution on the boundary. Since the boundary conditions are important in the integrations by parts leading to the energy estimates, we have to limit ourselves to differentiating with respect to time. This leads to energy estimates for time derivatives of the solution. Estimates for spatial derivatives must be obtained in a different fashion. For equations such as those of elasticity this is done by using elliptic regularity results to estimate spatial derivatives in terms of time derivatives (see [3],[4],[8]). In the present paper, we show how this method can be applied to the Dirichlet initial-boundary value problem for incompressible elasticity. We shall prove an existence theorem, but we do not address the question of whether or not our solution is the limit of solutions for compressible problems.

We consider a homogeneous hyperelastic body with reference configuration Ω and unit reference density. By x we denote material coordinates, and by $y(x,t)$ we denote the spatial position at time t of the particle with material coordinates x. We use the notation $F_\alpha^i = \partial y^i / \partial x^\alpha$ and we write $\partial x^\alpha / \partial y^i$ for the components of the inverse matrix F^{-1}. The equation of motion is as follows:

$$\ddot{y}^i = -\frac{\partial p}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^i} + \frac{\partial^2 W(F)}{\partial F_\alpha^i \partial F_\beta^j}\frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta} + f^i. \tag{1.1}$$

Here $W = W(F)$ is the stored energy function, $p = p(x,t)$ is an unknown pressure, and $f = f(x,t)$ is a prescribed body force. Throughout this paper we adopt the Einstein summation convention. The motion must satisfy the incompressibility constraint

$$\det F = 1. \tag{1.2}$$

We seek a solution to (1.1) and (1.2) subject to the initial conditions

$$y(x,0) = y_0(x), \ \dot{y}(x,0) = y_1(x), \tag{1.3}$$

and the Dirichlet boundary condition

$$y(x,t) = x \text{ for } t > 0, \ x \in \partial\Omega. \tag{1.4}$$

In order to make the pressure unique, we shall normalize it by

$$\int_\Omega p(x,t) \, dx = 0. \tag{1.5}$$

We make the following smoothness assumptions:
(S1) Ω is a bounded domain (i.e., an open, connected set) in R^3 with a boundary of class[1] $C^{3,1}$.

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[1] We could weaken this assumption and impose a "Sobolev-type" regularity on the boundary.

(S2) The stored energy function W is of class[2] C^5.

(S3) The initial data satisfy[3] $y_0 \in H^4(\Omega)$, $y_1 \in H^3(\Omega)$.

(S4) For some $T > 0$, we have

$$f \in \bigcap_{k=0}^{2} C^k([0,T]; H^{2-k}(\Omega)),$$

and the third order time derivative (in the sense of distributions) of f lies in $L^1([0,T]; L^2(\Omega))$.

Moreover, we assume that the elastic energy satisfies the strong ellipticity condition.

(E) With

$$A_{ij}^{\alpha\beta} = \frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}, \tag{1.6}$$

we have

$$A_{ij}^{\alpha\beta} \varsigma^i \varsigma^j \eta_\alpha \eta_\beta \geq C |\varsigma|^2 |\eta|^2 \tag{1.7}$$

for some positive constant C and all vectors $\varsigma, \eta \in R^3$.

Finally, we have to assume compatibility of the initial data with the boundary and incompressibility conditions. The following compatibility conditions are required.

(C1) $y_0 = x$ and $y_1 = 0$ on $\partial\Omega$.

(C2) $\det \nabla y_0 = 1$, and y_1 satisfies the following equation, which is obtained by differentiating the incompressibility condition with respect to time:

$$\frac{\partial y_1^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y_0^i} = 0. \tag{1.8}$$

(C3) The initial values of $\ddot{y}$ and $\partial^3 y / \partial t^3$, henceforth denoted by y_2 and y_3, vanish on $\partial\Omega$.

The last condition requires some explanation. The initial values for $\ddot{y}$ and $\partial^3 y / \partial t^3$ have to be determined from the differential equation. However, in order to find $\ddot{y}$ from the equation, we must first find the pressure at time $t = 0$. The procedure for this is as follows. First we differentiate the incompressibility condition once more with respect to time to obtain

$$\frac{\partial \ddot{y}^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} - \frac{\partial \dot{y}^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial \dot{y}^j}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^i} = 0. \tag{1.9}$$

From this, the quantity

$$\frac{\partial \ddot{y}^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} \tag{1.10}$$

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[2] We always think of W as being defined even for $\det F \neq 1$. This can always be achieved by extension.

[3] In our notation for Sobolev spaces, we do not distinguish the number of components of the function, i.e. $H^4(\Omega)$ may be a space of scalar-, vector- or matrix-valued functions depending on context.

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is known at time $t = 0$. If we now apply the "divergence operator" $(\partial x^\gamma / \partial y^i)(\partial / \partial x^\alpha)$ to the equation of motion (1.1), we obtain an elliptic differential equation for p, which has the form

$$\frac{\partial x^\gamma}{\partial y^i} \frac{\partial}{\partial x^\gamma} \left[\frac{\partial x^\alpha}{\partial y^i} \frac{\partial p}{\partial x^\alpha} \right] = g, \tag{1.11}$$

where g is known at time $t = 0$ and lies in $H^1(\Omega)$. On the boundary $\partial\Omega$, we multiply equation (1.1) by $(\partial x^\gamma / \partial y^i) n_\gamma$, where n is the outer unit normal. Since (1.4) implies that $\ddot{y}$ must vanish on $\partial\Omega$, the appropriate boundary condition for p has the form

$$\frac{\partial x^\gamma}{\partial y^i} n_\gamma \left[\frac{\partial x^\alpha}{\partial y^i} \frac{\partial p}{\partial x^\alpha} \right] = h, \tag{1.12}$$

where h is known at time $t = 0$ and lies in $H^{3/2}(\partial\Omega)$. Equations (1.11) and (1.12) form a Neumann-type boundary value problem for p (in fact, they are the Neumann problem if we transform to Eulerian coordinates). This problem has a unique solution $p \in H^3(\Omega)$ subject to the constraint (1.5), provided the right hand sides satisfy the condition

$$\int_\Omega g \, dx = \int_{\partial\Omega} h \, dS. \tag{1.13}$$

To verify that (1.13) holds we must show that the second term on the left-hand side of (1.9) has integral zero over Ω. If we transform to Eulerian coordinates, with u denoting to the velocity field, we obtain for the integral in question

$$\int_\Omega (\nabla_y u) : (\nabla_y u)^T \, dy = - \int_\Omega (u \cdot \nabla_y)(\operatorname{div}_y u) \, dy = 0. \tag{1.14}$$

After having solved for p at time $t = 0$, henceforth denoted p_0, we can insert it into (1.1) and compute $\ddot{y}$ at time $t = 0$. Differentiating (1.1) once with respect to time, we obtain in an analogous fashion a Neumann problem for $\dot{p}$, which we must solve in order to find $\partial^3 y / \partial t^3$ at $t = 0$. Condition (C3) means that the so computed initial data for $\ddot{y}$ and $\partial^3 y / \partial t^3$ must vanish on $\partial\Omega$.

We now state our main result.

Theorem:

Let the hypotheses (S1)-(S4), (E) and (C1)-(C3) be satisfied. Then there is a $T' \in (0, T]$ such that the problem (1.1)-(1.5) has a unique solution (y, p) on $\Omega \times [0, T']$ with the regularity property

$$y \in \bigcap_{k=0}^{4} C^{4-k}([0, T']; H^k(\Omega)), \quad p \in \bigcap_{k=2}^{4} C^{4-k}([0, T']; H^{k-1}(\Omega)).$$

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Remarks:

1. It follows from the Sobolev imbedding theorem that the solution satisfies $y \in C^2$, $p \in C^1$ and it is therefore a classical solution of the differential equations.

2. The result can be modified to accommodate Dirichlet boundary data other than $y = x$ provided appropriate smoothness and compatibility conditions are satisfied.

3. The result can be extended to viscoelastic materials for which the leading order terms in the differential equation are like those in the elastic case, e.g. K-BKZ materials (see [13]).

4. It follows from (1.2) and (1.4) that the mappings $x \to y(x,t)$ are actually globally invertible, see [2] and [12].

5. If more regularity of Ω, W and the data is assumed, and the appropriate additional compatibility conditions are satisfied, then the solution also has higher regularity.

2. Outline of procedure

The solution of (1.1)-(1.5) will be obtained by a contraction argument which involves solving a sequence of linear initial-boundary value problems. There are, of course, numerous ways to formulate (1.1)-(1.5) as a fixed point problem; for technical reasons the precise form of the fixed point problem is a very delicate issue. Since the incompressibility constraint is not in quasilinear form, we shall work with a differentiated version of (1.1)-(1.5). As noted in the introduction, we differentiate with respect to time because of the boundary conditions. We shall actually differentiate (1.1) twice and (1.2) three times. This yields a linear initial-boundary value problem for $\ddot{y}$ and $\ddot{p}$. Bounds for y, $\dot{y}$, p and $\dot{p}$ will be obtained via elliptic estimates.

Assuming that (y,p) is a sufficiently smooth solution of (1.1)-(1.5), we introduce the notations $u := \dot{y}$, $z := \ddot{y} - \lambda(y - x)$, $q := \dot{p}$, and $\phi := \ddot{p}$, where λ is a positive constant to be chosen later. The original system (1.1), (1.2) can now be written as follows:

$$z^i = -\frac{\partial p}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^i} + \frac{\partial^2 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j}\frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta} - \lambda(y^i - x^i) + f^i, \qquad (2.1)_1$$

$$\det(\nabla y) = 1. \qquad (2.1)_2$$

After differentiating once with respect to time, we obtain

$$\dot{z}^i = -\frac{\partial q}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^i} + \frac{\partial^2 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j}\frac{\partial^2 u^j}{\partial x^\alpha \partial x^\beta}$$

$$+ \frac{\partial p}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^k}\frac{\partial u^k}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^i} + \frac{\partial^3 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j \partial F_\gamma^k}\frac{\partial u^k}{\partial x^\gamma}\frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta} - \lambda u^i + \dot{f}^i, \qquad (2.2)_1$$

$$\frac{\partial u^i}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^i} = 0. \qquad (2.2)_2$$

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Differentiating once more with respect to time, we obtain

$$\ddot{z}^i + \frac{\partial \phi}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} - \frac{\partial^2 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j} \frac{\partial^2 z^j}{\partial x^\alpha \partial x^\beta}$$

$$= \lambda \frac{\partial^2 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j} \frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta} + 2 \frac{\partial q}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^k} \frac{\partial u^k}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^i}$$

$$-2 \frac{\partial p}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^l} \frac{\partial u^l}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial y^k} \frac{\partial u^k}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^i} + \frac{\partial p}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^k} \Big[\frac{\partial z^k}{\partial x^\beta} + \lambda \frac{\partial y^k}{\partial x^\beta} - \lambda \delta_\beta^k \Big] \frac{\partial x^\beta}{\partial y^i}$$

$$+2 \frac{\partial^3 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j \partial F_\gamma^k} \frac{\partial u^k}{\partial x^\gamma} \frac{\partial^2 u^j}{\partial x^\alpha \partial x^\beta} + \frac{\partial^3 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j \partial F_\gamma^k} \Big[\frac{\partial z^k}{\partial x^\gamma} + \lambda \frac{\partial y^k}{\partial x^\gamma} - \lambda \delta_\gamma^k \Big] \frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta}$$

$$+ \frac{\partial^4 W(\nabla y)}{\partial F_\alpha^i \partial F_\beta^j \partial F_\gamma^k \partial F_\delta^l} \frac{\partial u^k}{\partial x^\gamma} \frac{\partial u^l}{\partial x^\delta} \frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta} - \lambda z^i - \lambda^2 y^i + \lambda^2 x^i + \ddot{f}^i, \qquad (2.3)_1$$

$$\frac{\partial z^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} + \lambda \Big[\frac{\partial y^i}{\partial x^\alpha} - \frac{\partial x^i}{\partial x^\alpha} \Big] \frac{\partial x^\alpha}{\partial y^i} - \frac{\partial u^i}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^j} \frac{\partial u^j}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^i} = 0. \qquad (2.3)_2$$

In differentiating terms like $\partial x^\alpha / \partial y^i$ we have used the identity

$$\frac{\partial}{\partial t} F^{-1} = -F^{-1} \frac{\partial F}{\partial t} F^{-1}. \qquad (2.4)$$

Instead of using $(2.3)_2$ directly, we shall differentiate this equation once more with respect to time and apply some transformations to it. In doing this, we use the identity

$$\frac{\partial}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} = 0, \qquad (2.5)$$

which is obtained as follows. Employing (2.4) with $\partial/\partial t$ replaced by $\partial/\partial x^\alpha$, we find

$$\frac{\partial}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} = - \frac{\partial x^\alpha}{\partial y^j} \frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\beta}{\partial y^i} = -\text{tr} \, (F^{-1} \frac{\partial F}{\partial x^\beta}) \frac{\partial x^\beta}{\partial y^i}. \qquad (2.6)$$

It follows from (1.2) that

$$\text{tr} \, F^{-1} \frac{\partial F}{\partial x^\beta} = 0. \qquad (2.7)$$

Using (2.5), we can rewrite the first two terms in $(2.3)_2$ as

$$\frac{\partial}{\partial x^\alpha} \Big[z^i \frac{\partial x^\alpha}{\partial y^i} + \lambda (y^i - x^i) \frac{\partial x^\alpha}{\partial y^i} \Big]. \qquad (2.8)$$

After differentiating with respect to time, this expression becomes

$$\frac{\partial}{\partial x^\alpha} \Big[\dot{z}^i \frac{\partial x^\alpha}{\partial y^i} + \lambda u^i \frac{\partial x^\alpha}{\partial y^i} - [z^i + \lambda (y^i - x^i)] \frac{\partial x^\alpha}{\partial y^j} \frac{\partial u^j}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^i} \Big]. \qquad (2.9)$$

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For dealing with the last term in $(2.3)_2$, it is convenient to transform to Eulerian coordinates. In these coordinates the term is simply equal to

$$(\nabla_y u) : (\nabla_y u)^T = \text{div}_y \left[(u \cdot \nabla_y) u \right] \tag{2.10}$$

(∇_y and $\check{\ }$ denote Eulerian space and time derivatives, while $\frac{d}{dt} = \check{\ } + (u \cdot \nabla_y)$ denotes the material time derivative). In (2.8), we set $v = (u \cdot \nabla_y) u$ and take the material time derivative. We obtain

$$\frac{d}{dt} \text{div}_y \, v = \text{div}_y \left[\check{v} + (u \cdot \nabla_y) v - (v \cdot \nabla_y) u \right]. \tag{2.11}$$

To proceed further, we note the identity

$$\text{div}_y \left[(u \cdot \nabla_y) v \right] = \text{div}_y \left[u \, \text{div}_y \, v + (v \cdot \nabla_y) u \right]. \tag{2.12}$$

Using this, we find

$$\frac{d}{dt} \text{div}_y \, v = \text{div}_y \left[\check{v} + u \, \text{div}_y \, v \right] = \text{div}_y \left[(\check{u} \cdot \nabla_y) u + (u \cdot \nabla_y) \check{u} + u \, \text{div}_y \, v \right]$$

$$= \text{div}_y \left[2(\check{u} \cdot \nabla_y) u + u \, \text{div}_y \, v \right]$$

$$= \text{div}_y \left[2(\frac{du}{dt} \cdot \nabla_y) u - 2 \left[((u \cdot \nabla_y) u) \cdot \nabla_y \right] u + u \left[(\nabla_y u) : (\nabla_y u)^T \right] \right]. \tag{2.13}$$

By transforming this back to Lagrangian coordinates and putting it together with (2.9), we finally obtain

$$\frac{\partial x^\alpha}{\partial y^i} \frac{\partial}{\partial x^\alpha} \left\{ \dot{z}^i + \lambda u^i - 3 \left[z^j + \lambda(y^j - x^j) \right] \frac{\partial u^i}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^j} \right.$$

$$\left. + 2 u^k \frac{\partial x^\beta}{\partial y^k} \frac{\partial u^j}{\partial x^\beta} \frac{\partial x^\gamma}{\partial y^j} \frac{\partial u^i}{\partial x^\gamma} - u^i \frac{\partial u^j}{\partial x^\beta} \frac{\partial x^\beta}{\partial y^k} \frac{\partial u^k}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial y^j} \right\} = 0. \tag{2.14}$$

We now describe the basic iteration scheme. Given functions $y_{(n)}$, $z_{(n)}$, $p_{(n)}$ and $\phi_{(n)}$ on $\Omega \times [0, T]$, we determine $y_{(n+1)}$, $z_{(n+1)}$, $p_{(n+1)}$ and $\phi_{(n+1)}$ as follows. First, we set $u_{(n)} := \dot{y}_{(n)}$ and $q_{(n)} := \dot{p}_{(n)}$. For technical reasons we also need to introduce a function $\tilde{u}_{(n)}$ which is another approximation of u possessing better regularity properties than $u_{(n)}$. We set

$$\tilde{u}_{(n)} = \Pi \left(u_{(n)}, y_1(x) + \int_0^t z_{(n)}(x, \tau) + \lambda(y_{(n)}(x, \tau) - x) \, d\tau \right), \tag{2.15}$$

where Π is the projection operator from an appropriate product of Hilbert spaces onto the diagonal.

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We then determine $z_{(n+1)}$ and $\phi_{(n+1)}$ by solving a pair of equations related to $(2.3)_1$ and (2.14). To describe this pair of equations, it is convenient to denote the expression on the right-hand side of $(2.3)_1$ by

$$G(y, \nabla y, \nabla^2 y, u, \nabla u, \nabla^2 u, z, \nabla z, \tilde{f}, x).$$

We replace equation $(2.3)_1$ with

$$\ddot{z}^i_{(n+1)} = -\frac{\partial \phi_{(n+1)}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^i_{(n)}} + \frac{\partial^2 W(\nabla y_{(n)})}{\partial F^i_\alpha \partial F^j_\beta}\frac{\partial^2 z^j_{(n+1)}}{\partial x^\alpha \partial x^\beta}$$

$$+ G^i(y_{(n)}, \nabla y_{(n)}, \nabla^2 y_{(n)}, u_{(n)}, \nabla u_{(n)}, \nabla^2 u_{(n)}, z_{(n)}, \nabla z_{(n)}, \tilde{f}, x). \tag{2.16$_1$}$$

In the analogue of (2.14) we shall use $\tilde{u}_{(n)}$ rather than $u_{(n)}$. More precisely, we replace (2.14) with

$$\frac{\partial x^\alpha}{\partial y^i_{(n)}}\frac{\partial}{\partial x^\alpha}\left\{\dot{z}^i_{(n+1)} + \lambda\tilde{u}^i_{(n)} - 3\left[z^j_{(n)} + \lambda(y^j_{(n)} - x^j)\right]\frac{\partial \tilde{u}^i_{(n)}}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^j_{(n)}}\right.$$

$$\left. + 2\tilde{u}^k_{(n)}\frac{\partial x^\beta}{\partial y^k_{(n)}}\frac{\partial \tilde{u}^j_{(n)}}{\partial x^\beta}\frac{\partial x^\gamma}{\partial y^j_{(n)}}\frac{\partial \tilde{u}^i_{(n)}}{\partial x^\gamma} - \tilde{u}^i_{(n)}\frac{\partial \tilde{u}^j_{(n)}}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^k_{(n)}}\frac{\partial \tilde{u}^k_{(n)}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial y^j_{(n)}}\right\}$$

$$=: \frac{\partial x^\alpha}{\partial y^i_{(n)}}\frac{\partial}{\partial x^\alpha}\left\{\dot{z}^i_{(n+1)} - H^i(\tilde{u}, \nabla\tilde{u}, y, \nabla y, z, x)\right\} = 0. \tag{2.16$_2$}$$

If we impose the initial conditions

$$z_{(n+1)}(x, 0) = z_0(x) = y_2(x) - \lambda(y_0(x) - x), \quad \dot{z}_{(n+1)}(x, 0) = z_1(x) = y_3(x) - \lambda y_1(x), \tag{2.16$_3$}$$

the boundary conditions

$$z_{(n+1)} = 0 \text{ for } x \in \partial\Omega, \ t \geq 0, \tag{2.16$_4$}$$

and the normalization

$$\int_\Omega \phi_{(n+1)}(x, t)\, dx = 0, \tag{2.16$_5$}$$

the problem (2.16) can be solved uniquely for $z_{(n+1)}$ and $\phi_{(n+1)}$.

Finally, we obtain $y_{(n+1)}$ and $p_{(n+1)}$ by solving a nonlinear elliptic equation. For each $t \in [0, T']$, we consider the boundary value problem

$$-\frac{\partial p_{(n+1)}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^i_{(n+1)}} + \frac{\partial^2 W(\nabla y_{(n+1)})}{\partial F^i_\alpha \partial F^j_\beta}\frac{\partial^2 y^j_{(n+1)}}{\partial x^\alpha \partial x^\beta} - \lambda(y^i_{(n+1)} - x^i) - z^i_{(n+1)} - f^i. \tag{2.17$_1$}$$

$$\det \nabla y_{(n+1)} = 1. \tag{2.17$_2$}$$

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$$\int_\Omega p_{(n+1)} = 0, \qquad\qquad (2.17)_3$$

$$y_{(n+1)}(x,t) = x \text{ for } x \in \partial\Omega. \qquad\qquad (2.17)_4$$

Using the implicit function theorem and standard results for linear elliptic equations, we can solve this boundary value problem for $y_{(n+1)}$ and $p_{(n+1)}$. By formally differentiating (2.17) with respect to time, we can obtain estimates for $\dot{y}$, $\ddot{y}$, $\dot{p}$ and $\ddot{p}$. The formal differentiation with respect to time can be justified rigorously by taking appropriate difference quotients and passing to the limit.

To implement the procedure outlined above, we must solve linear initial-boundary value problems of the form

$$\ddot{z}^i = -\chi_i^\alpha \frac{\partial\phi}{\partial x^\alpha} + A_{ij}^{\alpha\beta} \frac{\partial^2 z^j}{\partial x^\alpha \partial x^\beta} + G^i, \qquad\qquad (2.18)_1$$

$$\chi_i^\alpha \left(\frac{\partial \dot{z}^i}{\partial x^\alpha} - \frac{\partial H^i}{\partial x^\alpha} \right) = 0, \qquad\qquad (2.18)_2$$

where G^i, H^i and the coefficients χ_i^α, $A_{ij}^{\alpha\beta}$ are given. The solution of (2.18) is discussed in section 4 and will be based on the Galerkin method. In section 3, we discuss the needed results for elliptic systems. In section 5, we use the contraction mapping theorem to show that the iteration converges. The limit will provide the solution we seek.

In the estimates derived in the following two sections it is important to keep track of the quantities on which the constants appearing in the estimates depend. In the remaining sections, we always regard Ω, W (and hence the constant in the strong ellipticity condition), the forcing term f and the initial data as given once and for all. We shall therefore not explicitly point out any dependence on these quantities.

3. Elliptic estimates

In the proof of our theorem, we shall have to solve linear elliptic systems of the form

$$-\chi_i^\alpha \frac{\partial\phi}{\partial x^\alpha} + A_{ij}^{\alpha\beta} \frac{\partial^2 z^j}{\partial x^\alpha \partial x^\beta} + B_{ij}^\alpha \frac{\partial z^j}{\partial x^\alpha} - \lambda z^i = R^i, \qquad\qquad (3.1)_1$$

$$\chi_i^\alpha \frac{\partial z^i}{\partial x^\alpha} = S. \qquad\qquad (3.1)_2$$

$$z = 0 \text{ on } \partial\Omega, \qquad\qquad (3.1)_3$$

$$\int_\Omega \phi \, dx = 0. \qquad\qquad (3.1)_4$$

For our purposes, it is particularly important to understand how the solution (z, ϕ) depends on the norms of the coefficients χ_i^α, $A_{ij}^{\alpha\beta}$, B_{ij}^α. For convenience we introduce a parameter K which controls the size of the coefficients.

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We make the following assumptions:

(S1*) The coefficients χ_i^α and $A_{ij}^{\alpha\beta}$ are in $H^2(\Omega)$, and B_{ij}^α is in $H^1(\Omega)$. Moreover, we have $\|\chi\|_2 + \|A\|_2 + \|B\|_1 \le K$.

(S2*) For some given integer k with $-1 \le k \le 1$, we have $R \in H^k(\Omega)$, $S \in H^{k+1}(\Omega)$.

(I) The coefficients χ_i^α are the components of F^{-1}, where F is the gradient of a globally invertible mapping $y(x)$ with $\det F = 1$.

(E) We have $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$ and the strong ellipticity condition (1.7) holds.

(C*) $\int_\Omega g\, dx = 0$.

The following lemma holds:

Lemma 1:

Let assumptions (S1), (S1), (S2*), (I), (E) and (C*) hold. If $\lambda > 0$ is chosen sufficiently large relative to K, then the problem (3.1) has a unique solution. We have $z \in H^{k+2}(\Omega)$, $\phi \in H^{k+1}(\Omega)$ and an estimate of the form*

$$\|z\|_{k+2} + \|\phi\|_{k+1} \le C(\|R\|_k + \|S\|_{k+1}) \tag{3.2}$$

holds. The constant C depends solely on K and λ.

The proof employs standard techniques in elliptic theory and we omit details. One first obtains the existence of a unique weak solution by using Gårding's inequality and a standard variational argument. Higher regularity is obtained by using the fact that the system (3.1) is elliptic in the sense of Agmon, Douglis and Nirenberg [1]. The regularity results as stated in [1] only apply to solutions which a priori satisfy $z \in H^2(\Omega)$, $p \in H^1(\Omega)$ (as stated in [1], the coefficients would also be required to be of class C^2, but it is clear from the proofs in [1] that this is not actually required, since $H^2(\Omega)$ is an algebra in three spatial dimensions). Regularity of weak solutions is proved by Giaquinta and Modica [6] in the case of traction boundary conditions. As they remark, their techniques can also be applied for Dirichlet conditions.

For future use, we note that if f, g and the coefficients (regarded as taking values in the appropriate Sobolev spaces) are continuous (or bounded measurable) functions of a parameter t, then this property is inherited by the solution.

We also have to consider the solution of the nonlinear system (2.1) (with the boundary condition (1.4) and the normalization (1.5)) for given z and f. At the initial time $t = 0$ we have the solution $y = y_0$, $p = p_0$ for $z = z_0$ (the initial value of z) and $f = f_0(x) = f(x, 0)$. By using lemma 1, the implicit function theorem and the regularity theory of Agmon, Douglis and Nirenberg [1], we obtain the following result.

Lemma 2:

Let $\lambda > 0$ be chosen sufficiently large (relative to the data for the original problem). If z and f lie in $H^1(\Omega)$ and $\|z - z_0\|_1$, $\|f - f_0\|_1$ are sufficiently small, then (2.1), (1.4), (1.5) has a solution $(y, p) \in H^3(\Omega) \times H^2(\Omega)$. Within a neighborhood of (y_0, p_0) in $H^3(\Omega) \times H^2(\Omega)$ this solution is unique and it depends smoothly on $z \in H^1(\Omega)$ and $f \in H^1(\Omega)$. If moreover $z \in H^2(\Omega)$, $f \in H^2(\Omega)$, then $y \in H^4(\Omega)$, $p \in H^3(\Omega)$ and we have an estimate of the form

$$\|y\|_4 + \|p\|_3 \le C(1 + \|z\|_2 + \|f\|_2), \tag{3.3}$$

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where the constant C depends only on λ, $\|z - z_0\|_1$ and $\|f - f_0\|_1$.

4. A linearized problem

We seek a solution to (2.18) which satisfies the boundary condition

$$z = 0 \text{ on } \partial\Omega, \tag{4.1}$$

initial conditions

$$z(x,0) = z_0(x), \quad \dot{z}(x,0) = z_1(x), \tag{4.2}$$

and the normalization

$$\int_\Omega \phi(x,t) \, dx = 0. \tag{4.3}$$

As in section 3, it is imperative to understand how certain norms of the solution depend on corresponding norms of the coefficients. We use two parameters K and L to measure the sizes of χ_i^α and $A_{ij}^{\alpha\beta}$. In addition, it is convenient to introduce a parameter U which measures the sizes of G and H.

We make the following assumptions:

(S1′) The coefficients χ_i^α and $A_{ij}^{\alpha\beta}$ lie in $W^{2,\infty}([0,T]; H^1(\Omega)) \cap W^{1,\infty}([0,T]; H^2(\Omega)) \cap L^\infty([0,T]; H^3(\Omega))$ with norms bounded by L. Their norms in $L^\infty([0,T]; H^2(\Omega))$ are bounded by K.

(S2′) $G \in C([0,T]; L^2(\Omega))$, $\dot{G} \in L^1([0,T]; L^2(\Omega))$ and the norms are bounded by U.

(S3′) $H \in W^{2,1}([0,T]; L^2(\Omega)) \cap L^1([0,T]; H^2(\Omega))$ and the norms are bounded by U.

(S4′) $z_0 \in H^2(\Omega)$, $z_1 \in H^1(\Omega)$.

(E) We have $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$ and the strong ellipticity condition (1.7) holds.

(C1′) z_0 and z_1 vanish on $\partial\Omega$.

(C2′) z_1 satisfies $(2.18)_2$.

(C3′) H vanishes on $\partial\Omega$.

(I) The coefficients χ_i^α are the components of $F^{-1}(t)$, where $F(t)$ is the gradient of a globally invertible mapping $y(x,t)$ with $\det F = 1$.

We note that it follows from (I) that

$$\frac{\partial}{\partial x^\alpha} \chi_i^\alpha = 0.$$

The goal of this section is the following lemma.

Lemma 3:

Let assumptions (S1), (S1′)-(S4′), (E), (C1′)-(C3′) and (I) hold. Then the initial-boundary value problem defined by (2.18) and (4.1)-(4.3) has a unique solution (z, ϕ) defined on $\Omega \times [0, T]$ with the regularity property

$$z \in \bigcap_{k=0}^{2} C^k([0,T]; H^{2-k}(\Omega)), \quad \phi \in C([0,T]; H^1(\Omega)).$$

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Moreover, the solution obeys the a priori estimate

$$E[z,\phi](t) \leq \Gamma(U,K,T,T\cdot L) + \Lambda(U,K,T,L)\int_0^t E[z,\phi](s)\ ds\ \forall t \in [0,T], \qquad (4.4)_1$$

and hence

$$E[z,\phi](T) \leq \Gamma(U,K,T,T\cdot L)\exp(T\cdot\Lambda(U,K,T,L)), \qquad (4.4)_2$$

where

$$E[z,\phi](t) := \max_{s\in[0,t]}\left(\sum_{k=0}^2 \left\|\frac{\partial^k z}{\partial t^k}(.,s)\right\|_{2-k}^2 + \|\phi(.,s)\|_1^2\right). \qquad (4.4)_3$$

Γ and Λ are functions which are bounded on bounded sets.

It is important to note that Γ depends on L only through the combination $T\cdot L$, and therefore Γ can be controlled for large L by making T small.

For the proof of the lemma, we first note that by subtracting from z the reference function

$$\int_0^t H(.,\tau)\ d\tau$$

we can reduce the problem to the case $H = 0$. From now on we shall therefore assume $H = 0$. In order to solve (2.18), we differentiate once more with respect to time and then apply a Galerkin method to the time differentiated equation. Differentiation of (2.18) with respect to time yields

$$\frac{\partial^3 z^i}{\partial t^3} = -\chi_i^\alpha\frac{\partial\dot\phi}{\partial x^\alpha} + A_{ij}^{\alpha\beta}\frac{\partial^2\dot z^j}{\partial x^\alpha\partial x^\beta}$$

$$-\dot\chi_i^\alpha\frac{\partial\phi}{\partial x^\alpha} + \dot A_{ij}^{\alpha\beta}\frac{\partial^2 z^j}{\partial x^\alpha\partial x^\beta} + \dot G^i, \qquad (4.5)_1$$

$$\chi_i^\alpha\frac{\partial\ddot z^i}{\partial x^\alpha} + \dot\chi_i^\alpha\frac{\partial\dot z^i}{\partial x^\alpha} = 0. \qquad (4.5)_2$$

Because of (I) the latter equation can be rewritten as

$$\frac{\partial}{\partial x^\alpha}\left(\chi_i^\alpha\ddot z^i + \dot\chi_i^\alpha\dot z^i\right) = 0. \qquad (4.5)_2^{\cdot}$$

Moreover, recall that F_α^i is the matrix inverse to χ_i^α. We can write $(4.5)_2$ in the form

$$\chi_i^\alpha\frac{\partial}{\partial x^\alpha}\left(\ddot z^i - \dot F_\beta^i\chi_j^\beta\dot z^j\right) = 0. \qquad (4.5)_2^{\cdot\cdot}$$

In order to solve (4.5), we must prescribe initial data for $\ddot z$, which need to be determined from $(2.18)_1$. We must therefore find ϕ at time $t = 0$. For this purpose, we apply the operator $\chi_i^\gamma\frac{\partial}{\partial x^\gamma}$ to $(2.18)_1$, and we multiply by $\chi_i^\gamma n_\gamma$ on the boundary $\partial\Omega$. In dealing with the term involving $\ddot z^i$, we use $(4.5)_2^{\cdot\cdot}$ and the fact that $\ddot z^i$ should vanish on the

boundary. The initial values of z and $\dot{z}$ are known. In this fashion, we obtain a problem of the following form for ϕ:

$$\chi_i^\gamma \frac{\partial}{\partial x^\gamma}\left[\chi_i^\alpha \frac{\partial \phi}{\partial x^\alpha}\right] = \chi_i^\gamma \frac{\partial h^i}{\partial x^\gamma}, \tag{4.6}_1$$

$$\chi_i^\gamma n_\gamma \left[\chi_i^\alpha \frac{\partial \phi}{\partial x^\alpha}\right] = \chi_i^\gamma n_\gamma h^i \text{ on } \partial\Omega. \tag{4.6}_2$$

When transformed to Eulerian coordinates, this is simply the Neumann problem. Subject to the constraint (4.3), there is a unique weak solution $\phi \in H^1(\Omega)$ for every $h \in L^2(\Omega)$ (this is known as the Hodge projection theorem). We therefore have a unique initial condition $z_2 \in L^2(\Omega)$ for $\tilde{z}$ which is consistent with the initial data (4.1) and boundary data (4.2).

Instead of considering $(4.5)_1$ directly, we shall deal with the equation

$$\frac{\partial^3 z^i}{\partial t^3} = -\chi_i^\alpha \frac{\partial \dot{\phi}}{\partial x^\alpha} + A_{ij}^{\alpha\beta} \frac{\partial^2 \dot{z}^j}{\partial x^\alpha \partial x^\beta}$$

$$-\dot{\chi}_i^\alpha \frac{\partial \tilde{\phi}}{\partial x^\alpha} + \dot{A}_{ij}^{\alpha\beta} \frac{\partial^2 \tilde{z}^j}{\partial x^\alpha \partial x^\beta} + \dot{G}^i, \tag{4.7}$$

where $\tilde{\phi}$ and $\tilde{z}$ are expressed in terms of $\tilde{z} - \lambda z$ by the following elliptic problem obtained from (2.18):

$$\tilde{z}^i - \lambda z^i = -\chi_i^\alpha \frac{\partial \tilde{\phi}}{\partial x^\alpha} + A_{ij}^{\alpha\beta} \frac{\partial^2 \tilde{z}^j}{\partial x^\alpha \partial x^\beta} - \lambda \tilde{z}^i + G^i, \tag{4.8}_1$$

$$\chi_i^\alpha(x,t) \frac{\partial \tilde{z}^i(x,t)}{\partial x^\alpha} = \int_0^t \dot{\chi}_i^\alpha(x,\tau) \frac{\partial \tilde{z}^i(x,\tau)}{\partial x^\alpha}\, d\tau + \chi_i^\alpha(x,0) \frac{\partial z_0^i(x)}{\partial x^\alpha}. \tag{4.8}_2$$

The latter equation has been obtained by integrating $(2.18)_2$ (with $g = 0$) with respect to time. Of course we shall eventually show that $\tilde{z} = z$ for the solution we construct, and hence (4.7) is equivalent to $(4.5)_1$, but the approximations we are using for $\tilde{z}$ and z will actually differ. It follows from the elliptic estimates discussed in the previous section that (4.8) (with the boundary condition $\tilde{z} = 0$ and the normalization $\int_\Omega \tilde{\phi}(x)\, dx = 0$) can be solved for $\tilde{z}$ and $\tilde{\phi}$ if λ is chosen appropriately. Moreover, an estimate of the form

$$\|\tilde{z}\|_{L^\infty([0,T];H^2(\Omega))} + \|\tilde{\phi}\|_{L^\infty([0,T];H^1(\Omega))}$$

$$\leq C\left(\|\tilde{z} - \lambda z\|_{L^\infty([0,T];L^2(\Omega))} + \|G\|_{L^\infty([0,T];L^2(\Omega))} + \|z_0\|_{H^2(\Omega)}\right) \tag{4.9}$$

holds.

Let now $\tilde{V}$ be the space of all divergence-free vector fields in $H_0^1(\Omega)$ and let $\{\xi_1, \xi_2, \xi_3, ...\}$ be a basis for $\tilde{V}$. Moreover, let $V(t) = F(t)\tilde{V} = \{w \in H_0^1(\Omega) \mid F^{-1}(t)w \in \tilde{V}\}$, and let $\eta_\kappa(t) = F(t)\xi_\kappa$. Let $\tilde{V}_n = \operatorname{span}\{\xi_1, \xi_2, ..., \xi_n\}$ and $V_n(t) = F(t)\tilde{V}_n$. We seek an approximate solution of the form

$$\dot{z}_n(t) = \sum_{\kappa=1}^n a_{n\kappa}(t)\eta_\kappa(t), \tag{4.10}$$

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(i.e., $\dot{z} \in V_n(t)$), which satisfies the following approximate version of (4.7)

$$\langle F_\gamma^i \omega^\gamma, \frac{\partial^3(z_n)^i}{\partial t^3} - A_{ij}^{\alpha\beta}\frac{\partial^2(\dot{z}_n)^j}{\partial x^\alpha \partial x^\beta} + \dot{\chi}_i^\alpha \frac{\partial \tilde{\phi}_n}{\partial x^\alpha} - \dot{A}_{ij}^{\alpha\beta}\frac{\partial^2(\tilde{z}_n)^j}{\partial x^\alpha \partial x^\beta} - \dot{G}^i \rangle = 0 \qquad (4.11)$$

for every $\omega \in \tilde{V}_n$. Here $\tilde{\phi}_n$ and $\tilde{z}_n$ are determined from (4.8) with z on the left side of $(4.8)_1$ replaced by z_n, and $\langle .,. \rangle$ denotes the inner product in $L^2(\Omega)$. Finally, we define initial data by

$$z_n(x,0) = z_{n0}(x) = z_0(x), \ \dot{z}_n(x,0) = z_{n1}(x) = P_n^1 z_1(x),$$

$$\ddot{z}_n(x,0) = z_{n2}(x) = \dot{F}(x,0)F^{-1}(x,0)z_{n1}(x) + P_n^0\Big(z_2(x) - \dot{F}(x,0)F^{-1}(x,0)z_{n1}(x)\Big). \quad (4.12)$$

Here P_n^1 is the projection in $V(0)$ onto the subspace $V_n(0)$ and P_n^0 is the projection in $L^2(\Omega)$ onto $V_n(0)$.

Equation (4.11) is a linear system of ordinary integro-differential equations for the coefficients $a_{n\kappa}(t)$, which can be solved by standard methods. Hence a solution to (4.11) with the initial data (4.12) always exists. To obtain an energy estimate for this solution, we set

$$\omega = F^{-1}\Big(\ddot{z}_n - \dot{F}F^{-1}\dot{z}_n\Big) = \sum_{\kappa=1}^n \dot{a}_{n\kappa}(t)\xi_\kappa \qquad (4.13)$$

and integrate from 0 to t. This yields

$$\frac{1}{2}\int_\Omega |\ddot{z}_n(x,t)|^2 + A_{ij}^{\alpha\beta}\frac{\partial(\dot{z}_n)^i}{\partial x^\alpha}\frac{\partial(\dot{z}_n)^j}{\partial x^\beta} \ dx$$

$$= \frac{1}{2}\|z_{n2}\|_{L^2(\Omega)}^2 + \langle \dot{F}(t)F^{-1}(t)\dot{z}_n(t), \ddot{z}_n(t)\rangle - \langle \dot{F}(0)F^{-1}(0)z_{n1}, z_{n2}\rangle$$

$$- \int_0^t \langle \ddot{F}(\tau)F^{-1}(\tau)\dot{z}_n(\tau) - \dot{F}(\tau)F^{-1}(\tau)\dot{F}(\tau)F^{-1}(\tau)\dot{z}_n(\tau) + \dot{F}(\tau)F^{-1}(\tau)\ddot{z}_n(\tau), \ddot{z}_n(\tau)\rangle d\tau$$

$$+ \frac{1}{2}\langle \frac{\partial(z_{n1})^i}{\partial x^\alpha}, A_{ij}^{\alpha\beta}\frac{\partial(z_{n1})^j}{\partial x^\beta}\rangle + \frac{1}{2}\int_0^t \langle \frac{\partial(\dot{z}_n)^i}{\partial x^\alpha}, \dot{A}_{ij}^{\alpha\beta}\frac{\partial(\dot{z}_n)^j}{\partial x^\beta}\rangle \ d\tau$$

$$- \int_0^t \langle (\ddot{z}_n)^i, \Big(\frac{\partial}{\partial x^\alpha}A_{ij}^{\alpha\beta}\Big)\frac{\partial(\dot{z}_n)^j}{\partial x^\beta}\rangle \ d\tau$$

$$+ \int_0^t \langle \frac{\partial}{\partial x^\alpha}\big(\dot{F}_\gamma^i \chi_k^\gamma(\dot{z}_n)^k\big), A_{ij}^{\alpha\beta}\frac{\partial(\dot{z}_n)^j}{\partial x^\beta}\rangle \ d\tau$$

$$+ \int_0^t \langle (\ddot{z}_n)^i - \dot{F}_\gamma^i \chi_k^\gamma(\dot{z}_n)^k, -\dot{\chi}_i^\alpha\frac{\partial\tilde{\phi}_n}{\partial x^\alpha} + \dot{A}_{ij}^{\alpha\beta}\frac{\partial^2(\tilde{z}_n)^j}{\partial x^\alpha \partial x^\beta} + \dot{G}^i\rangle \ d\tau. \qquad (4.14)$$

From this it follows that we have uniform bounds on the norms of $\ddot{z}_n$ in $L^\infty([0,T]; L^2(\Omega))$ and $\dot{z}_n$ in $L^\infty([0,T]; H^1(\Omega))$. After passing to a subsequence, we may therefore assume

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weak-∗ convergence in those spaces. An argument along the lines of Chapter III, §1 in [15] can be used to show that the limit z is in fact a solution of (4.7) and $(2.18)_2$. Obviously, $\tilde{z} \in L^\infty([0,T]; L^2(\Omega))$ and $\dot{z} \in L^\infty([0,T]; H^1(\Omega))$. Moreover, it can be shown that $\tilde{z}$ is weakly continuous into $L^2(\Omega)$ (and in that sense assumes the given initial data), that ϕ is in $L^\infty([0,T]; L^2(\Omega))$, and that

$$\frac{\partial^3 z^i}{\partial t^3} + \chi_i^\alpha \frac{\partial \dot{\phi}}{\partial x^\alpha}$$

lies in $L^\infty([0,T]; H^{-1}(\Omega))$. The latter need not be true for each of the two terms separately. The reason for this is that the Hodge decomposition can not be applied in H^{-1}. In this respect the initial-boundary value problem differs from the problem on all of space.

To see the weak continuity of $\tilde{z}$, we first note that $\langle \omega, F^T \tilde{z} \rangle$ is continuous for every solenoidal test function ω. Therefore, $PF^T \tilde{z}$ is weakly continuous, where P denotes the Hodge projection in $L^2(\Omega)$. Let now $\tilde{z} = Fv$. Then it follows from $(4.5)_2$ that

$$v - F^{-1}\dot{F}F^{-1}\dot{z} = P(v - F^{-1}\dot{F}F^{-1}\dot{z}), \tag{4.15}$$

and hence $v - Pv$ is continuous in $L^2(\Omega)$ as a function of time. As a consequence, $PF^T FPv$ is weakly continuous, and since the operator $PF^T FP$ is invertible on $PL^2(\Omega)$, Pv is weakly continuous. It follows that v and hence $\tilde{z}$ is weakly continuous.

In order to show that we actually obtain a solution of (2.18), we have to prove that $\tilde{\phi} = \phi$ and $\tilde{z} = z$. By integrating (4.7) with respect to time, we find

$$\ddot{z}^i = -\chi_i^\alpha \frac{\partial \phi}{\partial x^\alpha} + A_{ij}^{\alpha\beta} \frac{\partial z^j}{\partial x^\alpha \partial x^\beta} + G^i$$

$$+ \int_0^t -\dot{\chi}_i^\alpha \left(\frac{\partial \tilde{\phi}}{\partial x^\alpha} - \frac{\partial \phi}{\partial x^\alpha} \right) + \dot{A}_{ij}^{\alpha\beta} \left(\frac{\partial \tilde{z}^j}{\partial x^\alpha \partial x^\beta} - \frac{\partial z^j}{\partial x^\alpha \partial x^\beta} \right) \, d\tau. \tag{4.16}$$

In view of (4.8), we can rewrite this as

$$-\chi_i^\alpha \left(\frac{\partial \phi}{\partial x^\alpha} - \frac{\partial \tilde{\phi}}{\partial x^\alpha} \right) + A_{ij}^{\alpha\beta} \left(\frac{\partial^2 z^j}{\partial x^\alpha \partial x^\beta} - \frac{\partial \tilde{z}^j}{\partial x^\alpha \partial x^\beta} \right) - \lambda(z^i - \tilde{z}^i)$$

$$+ \int_0^t -\dot{\chi}_i^\alpha \left(\frac{\partial \tilde{\phi}}{\partial x^\alpha} - \frac{\partial \phi}{\partial x^\alpha} \right) + \dot{A}_{ij}^{\alpha\beta} \left(\frac{\partial \tilde{z}^j}{\partial x^\alpha \partial x^\beta} - \frac{\partial z^j}{\partial x^\alpha \partial x^\beta} \right) \, d\tau = 0. \tag{4.17}$$

Similarly, by integrating $(2.18)_2$ and using $(4.8)_2$ we obtain

$$\chi_i^\alpha \left(\frac{\partial \tilde{z}^i}{\partial x^\alpha} - \frac{\partial z^i}{\partial x^\alpha} \right) = \int_0^t \dot{\chi}_i^\alpha \left(\frac{\partial \tilde{z}^i}{\partial x^\alpha} - \frac{\partial z^i}{\partial x^\alpha} \right) \, d\tau. \tag{4.18}$$

It follows from the elliptic uniqueness result established in the previous section and a straightforward perturbation argument that $\tilde{z} = z$ and $\tilde{\phi} = \phi$.

It remains to be shown that $\tilde{z}$ is actually strongly continuous into $L^2(\Omega)$ and that $\dot{z}$ is strongly continuous into $H^1(\Omega)$. To do this, we first rewrite (4.7) in a different form

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by first multiplying the equation by F^T (this transforms the $\dot{\phi}$-term into $\nabla\dot{\phi}$) and then substituting $\dot{z} = Fw$ (this transforms the incompressibility condition into div $w = 0$. We can then apply the technique on pp. 276-279 of [11], which employs mollifiers with respect to time. This is why it is important to write the incompressibility constraint in a form which persists under such mollification.

Finally, to obtain the a priori estimate (4.4), we proceed along the lines of the derivation of (3.15) in [4]. The basic idea is that by virtue of the weak-$*$ lower semicontinuity of L^∞-type norms, (4.14) yields an energy inequality for z. Using this inequality together with (4.9) standard Sobolev estimates, and Gronwall's inequality, we obtain (4.4). See section 3 of [4] for the details in a similar situation. Of course, in the derivation of (4.4), we must account for the fact that we have subtracted the reference function $\int_0^t H(.,\tau)\,d\tau$ from z.

5. Convergence of the iteration

In this section we prove that the iteration outlined in section 2 converges in an appropriate space of functions. The iteration involves the quadruplet of functions (y, z, p, ϕ). Let $T > 0$ and $M > 0$ be given (later, we shall choose T small and M large). We denote by $\|y\|_{k,l}$ the norm of y in $W^{k,\infty}([0,T]; H^l(\Omega))$. By $Z(T, M)$ we denote the set of all (y, z, p, ϕ) which satisfy the following conditions:

$$\|y\|_{0,4}^2 + \|y\|_{1,3}^2 + \|y\|_{2,2}^2 \le M^2, \tag{5.1}_1$$

$$\|p\|_{0,3}^2 + \|p\|_{1,2}^2 + \|p\|_{2,1}^2 \le M^2, \tag{5.1}_2$$

$$\|z\|_{0,2}^2 + \|z\|_{1,1}^2 + \|z\|_{2,0}^2 \le M^2, \tag{5.1}_3$$

$$\|\phi\|_{0,1} \le M, \tag{5.1}_4$$

$$z(x,0) = z_0(x), \tag{5.2}_1$$

$$y(x,0) = y_0(x), \tag{5.2}_2$$

$$\det \nabla y = 1, \tag{5.3}$$

$$y = x \text{ on } \partial\Omega, \tag{5.4}_1$$

$$z = 0 \text{ on } \partial\Omega, \tag{5.4}_2$$

Clearly the set $Z(T, M)$ contains $(y_0, z_0, 0, 0)$ (provided M is chosen large enough), hence it is not empty. If $(y, z, p, \phi) \in Z(T, M)$, we have

$$u = \dot{y} \in L^\infty([0,T]; H^3(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$$

$$\subset L^2([0,T]; H^3(\Omega) \cap H_0^1(\Omega)) \cap H^1([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) =: X_1. \tag{5.5}_1$$

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On the other hand we have

$$y_1(x) + \int_0^t z(x,\tau) + \lambda(y(x,\tau) - x) \, d\tau$$

$$\in W^{1,\infty}([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{2,\infty}([0,T]; H_0^1(\Omega))$$

$$\subset H^1([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^2([0,T]; H_0^1(\Omega)) =: X_2. \tag{5.5}_2$$

We now choose Π in (2.15) to be the orthogonal projection of the Hilbert space $X_1 \times X_2$ onto the diagonal $X_1 \cap X_2$. Let Σ be the mapping defined by the procedure of section 2 that takes $(y_{(n)}, z_{(n)}, p_{(n)}, \phi_{(n)})$ to $(y_{(n+1)}, z_{(n+1)}, p_{(n+1)}, \phi_{(n+1)})$. It follows from the results of section 3 and 4 that Σ is well defined if M is large enough relative to the initial data, T is small enough relative to M and λ is sufficiently large. The reason why T must be small relative to M is to ensure that $\|z_{n+1}(.,t) - z_0\|_1$ remains small enough so that lemma 2 can be used to solve (2.17).

We shall show below that if M is increased and T is reduced accordingly, then $(y_{(n)}, z_{(n)}, p_{(n)}, \phi_{(n)})$ converges (in an appropriate sense) to an element (y, z, p, ϕ) of $Z(T, M)$ as $n \to \infty$. We shall then show that the pair (y, p) is a solution of (1.1)-(1.5). In the proofs of lemmas 4 and 5 we need to solve the nonlinear elliptic problem (2.17) as well as several linear elliptic problems of the form (3.1). The parameter λ must be large enough so that these problems can be solved. Examination of the proofs of lemmas 4 and 5 reveals that for each problem of the form (3.1), $\|\chi_i^\alpha\|_2 + \|A_{ij}^{\alpha\beta}\|_2 + \|B_{ij}^\alpha\|_1$ remains bounded as M gets large, provided T is small enough relative to M. It is therefore possible to select λ sufficiently large (relative to the data for the original problem (1.1)-(1.5)) so that no matter how large M is made, all of the elliptic problems occurring in the proofs of lemmas 4 and 5 can be solved provided T is small. We now fix such a λ once and for all. Moreover, from now on we shall always assume T is small enough (relative to M) so that Σ is well defined.

Lemma 4:

If M is chosen sufficiently large and T sufficiently small relative to M, then Σ maps the set $Z(T, M)$ into itself.

Proof:

Let $M, T > 0$ and $(y_{(n)}, z_{(n)}, p_{(n)}, \phi_{(n)})$ be given and set $(y_{(n+1)}, z_{(n+1)}, p_{(n+1)}, \phi_{(n+1)}) = \Sigma(y_{(n)}, z_{(n)}, p_{(n)}, \phi_{(n)})$. Let $u_{(n)}, \tilde{u}_{(n)}, q_{(n)}, G$ and H be as in section 2. We first apply lemma 3 to obtain an a priori estimate for $z_{(n+1)}$ and $\phi_{(n+1)}$. For this purpose we identify $A_{ij}^{\alpha\beta}$ with $\frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n)})$ and χ_i^α with $\frac{\partial x^\alpha}{\partial y_{(n)}^i}$. Observe that

$$\|A_{ij}^{\alpha\beta}\|_{0,2} \leq \|\frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_0)\|_2$$

$$+ \int_0^T \|\frac{\partial^3 W(\nabla y_{(n)})}{\partial F_\alpha^i \partial F_\beta^j \partial F_\gamma^k} \frac{\partial u_{(n)}^k}{\partial x^\gamma}(.,t)\|_2 \, dt, \tag{5.6}$$

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from which we conclude that $\|A\|_{0,2}$ is bounded by a constant plus T times a continuous function of M. A similar comment applies to $\|\chi_t^\alpha\|_{0,2}$. This means that these norms can be controlled for large M by choosing T small.

Next we note that the appropriate norms of G and H, i.e. the quantity called U in lemma 3, can be estimated by a function of M and T which stays bounded if T is chosen small enough relative to M. The reason for this is that the terms appearing in G and H actually have better temporal regularity than is required by lemma 3, and the desired bound can be obtained using Hölder's inequality. Employing the a priori estimate (4.4), we find that

$$E[z_{(n+1)}, \phi_{(n+1)}](T) \leq \Gamma^*(M, T) \exp(T \cdot \Lambda^*(M, T)), \tag{5.7}$$

where $\Gamma^*(M, T)$ remains bounded if T is chosen small enough relative to M and $\Lambda^*(M, T)$ is bounded for M and T in bounded sets. It follows from this that $z_{(n+1)}$ and $\phi_{(n+1)}$ satisfy $(5.1)_3$ and $(5.1)_4$ provided M is large and T is small.

To obtain estimates for $y_{(n+1)}$ and $p_{(n+1)}$ we proceed as outlined in section 2, making use of the results of section 3. Observe that

$$\|z_{(n+1)}(., t) - z_0\|_1 \leq \int_0^T \|\dot{z}_{(n+1)}(., \tau)\|_1 \, d\tau \leq MT \ \forall t \in [0, T]. \tag{5.8}$$

Thus if T is small enough relative to M we may use lemma 2 to solve (2.17) for $y_{(n+1)}$ and $p_{(n+1)}$; we also obtain an estimate for $\|y_{(n+1)}\|_4 + \|p_{(n+1)}\|_3$ by virtue of (3.3). Formal differentiation of (2.17) with respect to time yields an estimate for $\|\dot{y}_{(n+1)}\|_3 + \|\dot{p}_{(n+1)}\|_2$ by virtue of lemma 1. Of course, this procedure is not quite legitimate because we do not know a priori that $\dot{y}_{(n+1)}$ and $\dot{p}_{(n+1)}$ exist as functions. However, the estimate can be obtained rigorously through the use of difference quotients. This argument can be applied once more to obtain a bound for $\|\ddot{y}_{(n+1)}\|_2 + \|\ddot{p}_{(n+1)}\|_0$.

Putting all the pieces together, we conclude that

$$\|y_{(n+1)}\|_{0,4}^2 + \|y_{(n+1)}\|_{1,3}^2 + \|y_{(n+1)}\|_{2,2}^2 + \|p_{(n+1)}\|_{0,3}^2$$

$$+ \|p_{(n+1)}\|_{1,2}^2 + \|p_{(n+1)}\|_{2,1}^2 \leq \tilde{\Gamma}(M, T), \tag{5.9}$$

where $\tilde{\Gamma}$ remains bounded as M gets large, provided T is small enough relative to M. The lemma now follows from (5.9) and our previous estimate for $z_{(n+1)}$ and $\phi_{(n+1)}$. We note that if M is held fixed and T is reduced, then Σ will still map $Z(T, M)$ into itself.

On $Z(T, M)$, we define the following pseudometric:

$$d((y, z, p, \phi), (\hat{y}, \hat{z}, \hat{p}, \hat{\phi})) = \|y - \hat{y}\|_{0,3} + \|y - \hat{y}\|_{1,2} + \|p - \hat{p}\|_{0,2} + \|p - \hat{p}\|_{1,1}$$

$$+ \|z - \hat{z}\|_{0,1} + \|z - \hat{z}\|_{1,0}. \tag{5.10}$$

The convergence of the iteration scheme now follows from the following lemma.

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Lemma 5:

If M is chosen sufficiently large, and T sufficiently small relative to M, then the mapping $\Sigma\colon Z(T,M) \to Z(T,M)$ is a contraction with respect to the pseudometric d.

Proof:

We use the same notation as in the proof of lemma 4. In addition, we set $Y_{(n)} := y_{(n+1)} - y_{(n)}$, $Z_{(n)} := z_{(n+1)} - z_{(n)}$, $P_{(n)} := p_{(n+1)} - p_{(n)}$, $\Phi_{(n)} := \phi_{(n+1)} - \phi_{(n)}$, $G_{(n)} := G(y_{(n)}, \nabla y_{(n)}, \nabla^2 y_{(n)}, u_{(n)}, \nabla u_{(n)}, \nabla^2 u_{(n)}, z_{(n)}, \nabla z_{(n)}, \tilde f, x)$, $H_{(n)} := H(\tilde u_{(n)}, \nabla \tilde u_{(n)}, y_{(n)}, \nabla y_{(n)}, z_{(n)}, x)$. We now fix M large enough so that lemma 4 applies for sufficiently small T. In the sequel we always assume T is small enough for lemma 4 to hold, but we may have to choose T still smaller. Since M is now fixed, we suppress the dependence of all estimates on M. It is convenient to define

$$E_n^*(t) = \max_{s \in [0,t]} \left(\|Z_{(n)}\|_1^2 + \|\dot Z_{(n)}\|_0^2 + \|Y_{(n)}\|_3^2 + \|\dot Y_{(n)}\|_2^2 + \|P_{(n)}\|_2^2 + \|\dot P_{(n)}\|_1^2 \right)(s). \quad (5.11)$$

A simple calculation gives the following problem for $Z_{(n+1)}$.

$$\ddot Z_{(n+1)}^i = -\frac{\partial \Phi_{(n+1)}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y_{(n+1)}^i} + \frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n+1)}) \frac{\partial^2 Z_{(n+1)}^j}{\partial x^\alpha \partial x^\beta}$$

$$+ \frac{\partial \phi_{(n+1)}}{\partial x^\alpha} \left(\frac{\partial x^\alpha}{\partial y_{(n)}^i} - \frac{\partial x^\alpha}{\partial y_{(n+1)}^i} \right)$$

$$+ \left(\frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n+1)}) - \frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n)}) \right) \frac{\partial z_{(n+1)}^j}{\partial x^\alpha \partial x^\beta} + G_{(n+1)}^i - G_{(n)}^i, \quad (5.12)_1$$

$$\frac{\partial x^\alpha}{\partial y_{(n+1)}^i} \left(\frac{\partial \dot Z_{(n+1)}^i}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha}(H_{(n+1)}^i - H_{(n)}^i) \right)$$

$$+ \left(\frac{\partial x^\alpha}{\partial y_{(n+1)}^i} - \frac{\partial x^\alpha}{\partial y_{(n)}^i} \right) \left(\frac{\partial \dot z_{(n+1)}^i}{\partial x^\alpha} - \frac{\partial H_{(n)}^i}{\partial x^\alpha} \right) = 0, \quad (5.12)_2$$

$$Z_{(n+1)}(x,0) = \dot Z_{(n+1)}(x,0) = 0, \quad (5.12)_3$$

$$Z_{(n+1)}(x,t) = 0 \text{ for } x \in \partial\Omega, \quad (5.12)_4$$

$$\int_\Omega \Phi_{(n+1)}(x,t)\, dx = 0. \quad (5.12)_5$$

We now use (5.12) to obtain an estimate for $\|Z_{(n+1)}\|_1^2 + \|\dot Z_{(n+1)}\|_0^2$. We multiply $(5.12)_1$ by $\dot Z_{(n+1)}$ and integrate with respect to space and time, performing a number of integrations by parts and utilizing the remaining equations in (5.12).

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After a routine (but rather long) computation we find that

$$\|Z_{(n+1)}(.,t)\|_1^2 + \|\dot{Z}_{(n+1)}\|_0^2 \leq \Gamma(T)E_n^*(T) + \Lambda(T)\int_0^t E_{n+1}^*(s)\,ds, \qquad (5.13)$$

where Γ and Λ are continuous functions with $\Gamma(0) = 0$. Differences of the form $\tilde{U}_{(n)} := \tilde{u}_{(n+1)} - \tilde{u}_{(n)}$ can easily be estimated in terms of $\dot{Y}_{(n)}$ and $Z_{(n)}$ by virtue of (2.15) and the fact that Π is a projection operator. The reason why $E_n^*(T)$ rather than $E_n^*(t)$ appears on the right hand side of (5.13) is the nonlocal nature of the projection operator with respect to time.

In order to assist the reader, we give the details of a typical calculation used in the derivation of (5.13). In particular, we shall obtain a bound for the integral

$$I(t) := \int_0^t \langle \dot{Z}_{(n+1)}(.,\tau), \left(\frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n+1)}) - \frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n)}) \right) \frac{\partial z_{(n+1)}^j}{\partial x^\alpha \partial x^\beta}(.,\tau) \rangle \, d\tau. \qquad (5.14)$$

To simplify the notation, let us set

$$\Psi_{ij}^{\alpha\beta}(x,\tau) := \left(\frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n+1)}) - \frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n)}) \right)(x,\tau). \qquad (5.15)$$

We first apply the Cauchy-Schwarz inequality to the integrand in $I(t)$. Next, we observe that

$$\|\Psi_{ij}^{\alpha\beta} \frac{\partial^2 z_{(n+1)}^j}{\partial x^\alpha \partial x^\beta}\|_0 \leq C\|\Psi\|_2 \|z_{(n+1)}\|_2 \leq CM\|\Psi\|_2 \qquad (5.16)$$

by standard Sobolev inequalities and the fact that $\|z_{(n+1)}\|_2 \leq M$. It follows from the assumed smoothness of W and standard results on composite mappings that

$$\|\Psi\|_2 \leq C_M\|\nabla Y_{(n)}\|_2 \leq C_M\|Y_{(n)}\|_3, \qquad (5.17)$$

where C_M is a constant depending on M. Employing (5.16), (5.17) and the elementary algebraic inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$, we find that

$$|I(t)| \leq \frac{1}{2} \int_0^t \|\dot{Z}_{(n+1)}(.,\tau)\|_0^2 \, d\tau$$

$$+ C\int_0^t \|Y_{(n)}\|_3^2 \, d\tau \leq C\left(TE_n^*(T) + \int_0^t E_{n+1}^*(\tau)\,d\tau \right) \forall t \in [0,T]. \qquad (5.18)$$

Since M has already been fixed, we now suppress the dependence of C on M in (5.18).

To obtain a bound for $\|Y_{(n+1)}\|_3 + \|P_{(n+1)}\|_2$, we use the elliptic problem

$$-\frac{\partial x^\alpha}{\partial y_{(n+2)}^i}\frac{\partial P_{(n+1)}}{\partial x^\alpha} + \frac{\partial^2 W}{\partial F_\alpha^i \partial F_\beta^j}(\nabla y_{(n+2)})\frac{\partial^2 Y_{(n+1)}^j}{\partial x^\alpha \partial x^\beta} - \lambda Y_{(n+1)}^i$$

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$$= Z^i_{(n+1)} + \frac{\partial p_{(n+1)}}{\partial x^\alpha}\left(\frac{\partial x^\alpha}{\partial y^i_{(n+2)}} - \frac{\partial x^\alpha}{\partial y^i_{(n+1)}}\right) + \left(\frac{\partial^2 W}{\partial F^i_\alpha \partial F^j_\beta}(\nabla y_{(n+1)})\right.$$

$$\left. - \frac{\partial^2 W}{\partial F^i_\alpha \partial F^j_\beta}(\nabla y_{(n+2)})\right)\frac{\partial^2 y^j_{(n+1)}}{\partial x^\alpha \partial x^\beta}, \tag{5.19}_1$$

$$\det \nabla y_{(n+2)} - \det \nabla y_{(n+1)} = 1, \tag{5.19}_2$$

$$Y_{n+1}(x,t) = 0 \text{ for } x \in \partial\Omega, \tag{5.19}_3$$

$$\int_\Omega P_{(n+1)}\, dx = 0. \tag{5.19}_4$$

The terms which are expressed as differences in $(5.19)_1$ can be rewritten in the form $B^\alpha_{ij}\frac{\partial Y^j}{\partial x^\alpha}$, using the identity

$$V(\nabla y_{(n+2)}) - V(\nabla y_{(n+1)}) = \left(\int_0^1 \nabla V(\nabla y_{(n+1)} + \eta \nabla Y_{(n+1)})\, d\eta\right)\cdot \nabla Y_{(n+1)}, \tag{5.20}$$

which holds for any smooth function V. A similar procedure can be applied to $(5.19)_2$. Applying lemma 1 to (5.19), we obtain a bound for $\|Y_{(n+1)}\|_3 + \|P_{(n+1)}\|_2$ in terms of $\|Z_{(n+1)}\|_1$. Similarly, by differentiating (5.19) with respect to time, we obtain a bound for $\|\dot{Y}_{(n+1)}\|_2 + \|\dot{P}_{(n+1)}\|_1$.

Combining all of these estimates and using Gronwall's inequality, we find that

$$E^*_{n+1}(T) \le \Gamma^*(T)\exp(T \cdot \Lambda^*(T))E^*_n(T), \tag{5.21}$$

where Γ^* and Λ^* are continuous functions with $\Gamma^*(0) = 0$. It follows from this that Σ is strictly contractive with respect to d if T is sufficiently small.

From lemma 5 it follows immediately that $y_{(n)}$, $z_{(n)}$ and $p_{(n)}$ converge in the metric given by d. We can then conclude directly from $(2.16)_1$, $(2.16)_5$ that $\phi_{(n)}$ converges in the sense of distributions and consequently in the weak-$*$ topology of $L^\infty([0,T];H^1(\Omega))$. Let (y,z,p,ϕ) be the limit of $(y_{(n)},z_{(n)},p_{(n)},\phi_{(n)})$ and let $\tilde{u}$ be the limit of $\hat{u}_{(n)}$. It is obvious that (y,z,p,ϕ) satisfies (2.16) and (2.17) (with the indices n and $n+1$ left out). To verify that we have in fact a solution of the original problem (1.1)-(1.5), it remains to be checked that $\tilde{y} = z + \lambda(y - x)$ (this implies as a consequence that $\hat{u} = u$) and $\tilde{p} = \phi$. For this purpose, we first note that by differentiating $(2.17)_1$ twice with respect to time we obtain $(2.16)_1$ with z replaced by $\tilde{y} - \lambda(y - x)$ and ϕ replaced by $\tilde{p}$ on the right hand side. Next we integrate $(2.16)_2$ with respect to time. This yields

$$\frac{\partial x^\alpha}{\partial y^i}\frac{\partial z^i}{\partial x^\alpha} = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial z^i_0}{\partial x^\alpha}$$

$$+ \int_0^t \frac{\partial x^\alpha}{\partial y^j}\frac{\partial u^j}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^i}\frac{\partial z^i}{\partial x^\alpha} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial}{\partial x^\alpha}H^i(\tilde{u}, \nabla \tilde{u}, y, \nabla y, z, x)\, d\tau. \tag{5.22}$$

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By differentiating $(2.17)_2$ twice with respect to time, we obtain the same equation with z replaced by $\ddot{y} - \lambda(y - x)$ and $\tilde{u}$ replaced by u. The next lemma yields an estimate for $\dot{u} - u$ in terms of $z - \ddot{y} + \lambda(y - x)$.

Lemma 6:

Let X_1, X_2 and Π be defined as in (5.5) above. Then there is a constant C such that for every $u \in X_1$ and $v \in X_2$ we have

$$\|u - \Pi(u,v)\|_{X_1} + \|v - \Pi(u,v)\|_{X_2} \leq C\|u - v\|_{H^1([0,T];H^2(\Omega) \cap H_0^1(\Omega))}. \tag{5.23}$$

For the proof, we note that it follows from the definition of Π that

$$\|u - \Pi(u,v)\|_{X_1}^2 + \|v - \Pi(u,v)\|_{X_2}^2 = \min_{u-v=r+s}\|r\|_{X_1}^2 + \|s\|_{X_2}^2 = \|u-v\|_{X_1 \dotplus X_2}^2. \tag{5.24}$$

Hence we want to show that the norm in $X_1 \dotplus X_2$ is equivalent to that in $H^1([0,T];H^2(\Omega) \cap H_0^1(\Omega))$. By the open mapping theorem, it is sufficient to show that the two spaces are equal in the set theoretic sense, i.e. that $H^1([0,T];H^2(\Omega) \cap H_0^1(\Omega))$ is contained in $X_1 \dotplus X_2$ (the reverse inclusion is trivial).

Let Δ be the Laplace operator on Ω with Dirichlet boundary conditions and $A = -\Delta$. We have $H_0^1(\Omega) = D(A^{1/2})$, $H^2(\Omega) \cap H_0^1(\Omega) = D(A)$, and $H^3(\Omega) \cap H_0^1(\Omega) \supset D(A^{3/2})$. Hence

$$Y_1 = L^2([0,T];D(A^{3/2})) \cap H^1([0,T];D(A)) \subset X_1, \tag{$5.25)_1$}$$

$$Y_2 = H^1([0,T];D(A)) \cap H^2([0,T];D(A^{1/2})) = X_2, \tag{$5.25)_2$}$$

and it suffices to show that $H^1([0,T];D(A)) \subset Y_1 \dotplus Y_2$. By using appropriate extension operators, we can further reduce this question to the analogous problem for temporally periodic functions (with any given period larger than T) rather than functions on the interval $[0,T]$. Hence let

$$W_1 = L_p^2([0,2\pi];D(A^{3/2})) \cap H_p^1([0,2\pi];D(A)), \tag{$5.26)_1$}$$

$$W_2 = H_p^1([0,2\pi];D(A)) \cap H_p^2([0,2\pi];D(A^{1/2})), \tag{$5.26)_2$}$$

$$W = H_p^1([0,2\pi];D(A)). \tag{$5.26)_3$}$$

The subscript p indicates that we are dealing with periodic functions.

We want to show that $W \subset W_1 \dotplus W_2$. Let ϕ_k denote the normalized eigenfunctions of A and λ_k^2 the corresponding eigenvalues. For each $w \in W$, we have an expansion

$$w = \sum_{k,l} w_{kl}\phi_k(x)e^{ilt}, \tag{5.27}$$

and

$$\|w\|_W = \sum_{k,l}(l^2 + 1)\lambda_k^4 |w_{kl}|^2. \tag{5.28}$$

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The desired decomposition is now obtained by setting

$$w_{kl}^1 = \frac{l^2 + 1}{\lambda_k^2 + l^2 + 1} w_{kl}, \ w_{kl}^2 = \frac{\lambda_k^2}{\lambda_k^2 + l^2 + 1} w_{kl}. \tag{5.29}$$

In particular, lemma 6 yields

$$\|\tilde{u} - u\|_{L^2([0,T];H^3(\Omega))} \le C\|\tilde{y} - z - \lambda(y - x)\|_{L^2([0,T];H^2(\Omega))}. \tag{5.30}$$

In conjunction with the uniqueness statement in lemma 1 and a standard perturbation argument, this implies that $z = \tilde{y} - \lambda(y - x)$ and $\phi = \tilde{p}$ as claimed.

Finally, the uniqueness of the solution of (1.1)-(1.5) follows from a calculation similar to the proof of lemma 5.

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